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PURE STRATEGY NASH EQUILIBRIA REFINEMENT IN BIMATRIX GAMES BY USING DOMINATION EFFICIENCY ALONG WITH MAXIMIN AND THE SUPEROPTIMALITY RULE

Background. Multiple Nash equilibria bring a new problem of selecting amongst them but this problem is solved by refining the equilibria. However, none of the existing refinements can guarantee a single refined Nash equilibrium. In some games, Nash equilibria are nonrefinable.
Objective. For solving the nonrefinability problem of pure strategy Nash equilibria in bimatrix games, the goal is to develop an algorithm which could facilitate in refining the equilibria as much further as possible.
Methods. A Nash equilibria refinement is suggested, which is based on the classical refinement by selecting only efficient equilibria that dominate by their payoffs. The suggested refinement exploits maximin subsequently. The superoptimality rule is involved if maximin fails to produce just a single refined equilibrium.
Results. An algorithm of using domination efficiency along with maximin and the superoptimality rule has been developed for refining Nash equilibria in bimatrix games. The algorithm has 10 definite steps at which the refinement is progressively accomplished. The developed concept of the equilibria refinement does not concern games with payoff symmetry and mirror-like symmetry.
Conclusions. The suggested pure strategy Nash equilibria refinement is a contribution to the equilibria refinement game theory field. The developed algorithm allows selecting amongst nonrefinable Nash equilibria in bimatrix games. It partially removes the uncertainty of equilibria, without going into mixed strategies. There are only two negative cases when the refinement fails. For a case when more than a single refined equilibrium is produced, the superoptimality rule may be used for a player having multiple refined equilibrium strategies but the other player has just a single refined equilibrium strategy.

Keywords: bimatrix game; Nash equilibria; refinement; domination efficiency; maximin; superoptimality rule.

Introduction

The Nash equilibrium is a solution concept of a noncooperative game, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only one's own strategy [1, 2]. Such a solution is used in a lot of fields involved "truly-thinking" objects that interact by having a few or more ways to do so. A classical example is economic/bioecologic interaction amongst competitors [3, 4].

A set of Nash equilibrium situations can be of more than just a single situation included each player's Nash equilibrium strategy. Such set, if non-singleton, is called Nash equilibria. Finding these equilibria (in pure strategies) can be a very hard task for infinite noncooperative games [1], but it is a very easy routine in finite noncooperative games, especially in bimatrix games [1, 5, 6].

As multiple Nash equilibria bring us a new problem of selecting amongst them, this problem is solved by refining the equilibria. Typically, such a refinement refers to the selection of a subset of Nash equilibria, and this subset is believed to include equilibria that are more plausible than other equilibria. A great deal of the refinements exists, e. g. [7], strong Nash equilibrium [8, 9], Mertens-stable equilibrium [10], trembling hand perfect equilibrium [11], proper equilibrium [12, 13], sequential equilibrium [14, 15], quasi-perfect equilibrium [13, 16, 17]. However, none of them can guarantee a single (refined) Nash equilibrium.

The refinement is turned on if Nash equilibria are obtained in a game. Seemingly, if the players' payoffs are the same for two or more Nash equilibria then these equilibria should not be refined. For instance, in a bimatrix game with payoff matrices

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \text{ by } a > 0,\ b > 0, \qquad (1)$$

we have two Nash equilibria wherein the payoffs remain the same – $\{a, b\}$ (we assume that the matrices are different). To stay at the equilibrium, the players must select different pure strategies (by their number-tags). And, as always, they will do that independently (and simultaneously). Nonetheless, what if the players select simultaneously just their first/second strategies? Surely, they will fall

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out of the equilibrium, and obtain zero payoffs. Obviously, here any refinement is helpless. The players in such a dyadic game are advised to get into a mixture of their Nash equilibria, whereupon the equilibria are selected randomly, with some probability (in this example, the selection is equiprobable).

A more sophisticated example is another dyadic game with different payoff matrices

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \qquad (2)$$

by the same parameters, where we have two Nash equilibria at the same strategies as previously exampled, but the payoffs are mirror-like – $\{a, b\}$ and $\{b, a\}$. It is naively clear that the probability of selecting the first pure strategy is $\frac{a}{a+b}$ for both the players, and $\frac{b}{a+b}$ is the probability of selecting their second pure strategies. But how knowing these probabilities helps? If the game is not to be repeated for at least a few tens of rounds [18, 19], then even sticking to those probabilities will not give us the expected payoffs, which are the same here being equal to $\frac{ab}{a+b}$. For the non-repeatable game, only Nash equilibria in pure strategies make sense [1, 2, 6, 20, 21]. But should we refine them here, as this mirror-like symmetry makes them apparently nonrefinable?

The example above shows a case of the Nash equilibria nonrefinability. No admissible decision rule is applicable here because there is no domination amongst the players' payoffs. Factually, those exampled payoffs are themselves efficient (being Pareto optimal). Thus the main problem is in the nonrefinability by using the known standard rules/concepts. This problem grows bigger in bimatrix games of greater dimensions, e. g. a bimatrix game with payoff matrices

$$\mathbf{A} = (a_{kj})_{3\times 6} = \begin{pmatrix} 5 & 2 & 4 & 0 & 2 & 3 \\ 5 & 5 & 3 & 4 & 1 & 5 \\ 2 & 6 & 3 & 0 & 6 & 2 \end{pmatrix}$$

and

$$\mathbf{B} = (b_{kj})_{3\times 6} = \begin{pmatrix} 3 & 4 & 6 & 5 & 4 & 4 \\ 4 & 2 & 3 & 3 & 4 & 0 \\ 3 & 3 & 0 & 2 & 3 & 3 \end{pmatrix} \qquad (3)$$

has four Nash equilibria, which are nonrefinable by the known concepts. Indeed, if we denote the players' pure strategy sets by $X = \{x_k\}_{k=1}^{3}$ and $Y = \{y_j\}_{j=1}^{6}$ then situations $\{x_1, y_3\}$, $\{x_2, y_1\}$, $\{x_3, y_2\}$, $\{x_3, y_5\}$ are the equilibria, at which the players' payoffs are $\{4, 6\}$, $\{5, 4\}$, $\{6, 3\}$, $\{6, 3\}$, respectively, without any domination amongst them. When we have more than just two players, the nonrefinability grows severer [1, 22, 23].

Problem statement

For solving the mentioned nonrefinability problem of pure strategy Nash equilibria in bimatrix games, the goal is to develop an algorithm which could facilitate in refining the equilibria as much further as possible. For achieving the goal, the five tasks are to be accomplished:

1. To state denotations that are going to be used.

2. To suggest an additional reasoning rule of selecting amongst nonrefinable Nash equilibria in bimatrix games.

3. To draw a scheme of an algorithm of using that rule for refining Nash equilibria as much further as possible.

4. To show illustratively how the algorithm works on real-valued examples of bimatrix games having Nash equilibria nonrefinable by the known concepts.

5. To discuss the developed algorithm and emphasize some unsolved issues, if any, in it.

The said task of showing illustratively the algorithm's work is aimed at understanding the process of refinement better. Advantages of selected/refined Nash equilibria should be seen clearly. Disadvantages and issues ought to be explained as well.

Denotations

We consider a bimatrix game with real-valued payoff matrices

$$\mathbf{A} = (a_{kj})_{M\times N} \text{ and } \mathbf{B} = (b_{kj})_{M\times N} \qquad (4)$$

of the first and second players, whose sets of pure strategies are $X = \{x_k\}_{k=1}^{M}$ and $Y = \{y_j\}_{j=1}^{N}$, $M \in \mathbb{N}\backslash\{1\}$, $N \in \mathbb{N}\backslash\{1\}$, respectively. The game is assumed to be non-repeatable. If $E = \{z_q\}_{q=1}^{Q}$ is a set of pure strategy Nash equilibria in bimatrix game (4), where $Q \leqslant M \cdot N$ (a number of the

pure strategy equilibria cannot exceed the total number of situations in pure strategies), then

$$z_q = \{x_{k^*}, y_{j^*}\} \text{ by } k^* \in K_* \subset \{\overline{1, M}\} \text{ and } j^* \in J_* \subset \{\overline{1, N}\}. \quad (5)$$

Recall that each element (5) of the Nash equilibria set satisfies inequalities

$$a_{kj^*} \leqslant a_{k^*j^*} \text{ and } b_{k^*j} \leqslant b_{k^*j^*} \ \forall k = \overline{1, M} \text{ and } \forall j = \overline{1, N}. \quad (6)$$

Thus subsets

$$X_* = \{x_{i^*}\}_{i^* \in K_*} \subset X \text{ and } Y_* = \{y_{l^*}\}_{l^* \in J_*} \subset Y \quad (7)$$

are formed. Therefore, for every element of set $X_* = \{x_{i^*}\}_{i^* \in K_*}$ $\exists q \in \{\overline{1, Q}\}$ such that $x_{i^*} \in z_q$, and for every element of set $Y_* = \{y_{l^*}\}_{l^* \in J_*}$ $\exists q \in \{\overline{1, Q}\}$ such that $y_{l^*} \in z_q$. Note that $E \subseteq X_* \times Y_*$ but every element of set $X_* \times Y_*$ is not necessarily an equilibrium point, i. e. some pairs $\{x_{i^*}, y_{l^*}\} \in \in X_* \times Y_*$ may not be the equilibria (see Fig. 1).

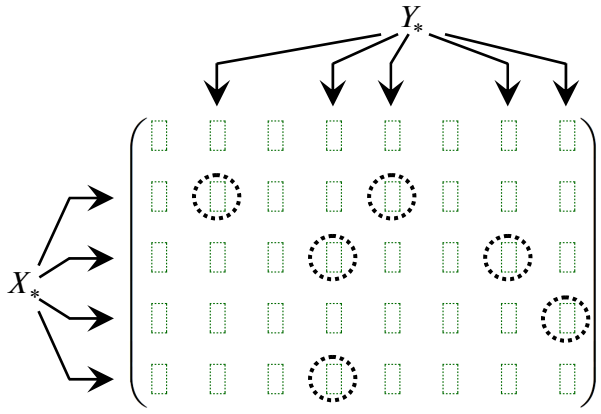

Fig. 1. An example sketch of the Nash equilibria set (highlighted via dashed circles) and its relation to subsets (7) over a player's payoff matrix

A set of efficient Nash equilibria is [1, 11, 12, 24, 25]

$$\hat{E} = \{\hat{z}_s\}_{s=1}^{S} \subset E \text{ by } S \leqslant Q, \quad (8)$$

where

$$\hat{z}_s = \{x_{k^{**}}, y_{j^{**}}\} \text{ for } k^{**} \in K_{**} \subset K_* \subset \{\overline{1, M}\} \text{ and } j^{**} \in J_{**} \subset J_* \subset \{\overline{1, N}\} \quad (9)$$

by subsets

$$\hat{X}_* = \{x_{i^{**}}\}_{i^{**} \in K_{**}} \subset X_* \subset X \text{ and } \hat{Y}_* = \{y_{l^{**}}\}_{l^{**} \in J_{**}} \subset Y_* \subset Y, \quad (10)$$

wherein for every element of set $\hat{X}_* = \{x_{i^{**}}\}_{i^{**} \in K_{**}}$ $\exists s \in \{\overline{1, S}\}$ such that $x_{i^{**}} \in \hat{z}_s$, and for every element of set $\hat{Y}_* = \{y_{l^{**}}\}_{l^{**} \in J_{**}}$ $\exists s \in \{\overline{1, S}\}$ such that $y_{l^{**}} \in \hat{z}_s$. Set (8) of efficient Nash equilibria is only of those points (9), at which neither a couple of inequalities

$$a_{k^*j^*} \geqslant a_{k^{**}j^{**}} \text{ and } b_{k^*j^*} > b_{k^{**}j^{**}} \quad (11)$$

nor a couple of inequalities

$$a_{k^*j^*} > a_{k^{**}j^{**}} \text{ and } b_{k^*j^*} \geqslant b_{k^{**}j^{**}} \quad (12)$$

is possible $\forall k^* \in K_*$ and $\forall j^* \in J_*$. Obviously,

$$\hat{E} \subset \hat{X}_* \times \hat{Y}_* \subset X_* \times Y_*.$$

Fig. 2 sketches out an example of these relations amongst sets E, (7), (8), (10).

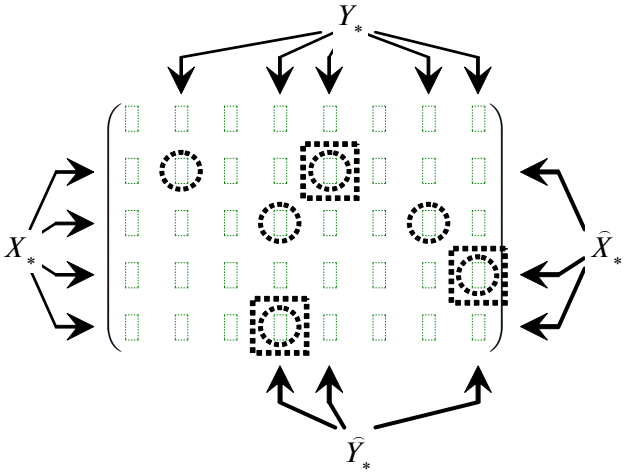

Fig. 2. An example sketch of relations amongst sets E, (7), (8), (10), where efficient Nash equilibria are via dashed rectangles

The refinement is not needed if $S = 1$. If $|\hat{X}_*| = 1$ then it is not needed also as set $\hat{X}_*$ contains an equilibrium strategy and the second player will use such strategy $y_{l^{**}} \in \hat{Y}_*$ at which its payoff is maximal. Analogously, if $|\hat{Y}_*| = 1$ then set $\hat{Y}_*$ contains an equilibrium strategy and the first player

will use such strategy $x_{i^{**}} \in \hat{X}_*$ at which its payoff is maximal.

Otherwise, before suggesting an additional reasoning rule of selecting amongst nonrefinable Nash equilibria in bimatrix games, a reduced bimatrix game defined on product $\hat{X}_* \times \hat{Y}_*$ is built. In such a game, payoff matrices are

$$\hat{\mathbf{A}} = (\hat{a}_{mn})_{M_{**} \times N_{**}} \text{ and } \hat{\mathbf{B}} = (\hat{b}_{mn})_{M_{**} \times N_{**}} \qquad (13)$$

where

$$M_{**} = |K_{**}| = |\hat{X}_*| > 1, \; N_{**} = |J_{**}| = |\hat{Y}_*| > 1,$$

and matrices (13) are the corresponding submatrices of (4) by $\hat{a}_{mn} = a_{k_m j_n}$, $\hat{b}_{mn} = b_{k_m j_n}$, with indices' sets $K_{**} = \{k_m\}_{m=1}^{M_{**}}$ and $J_{**} = \{j_n\}_{n=1}^{N_{**}}$.

Appending the maximin and superoptimality rule

The refinement must be turned on if $S > 1$. In the being considered non-repeatable bimatrix game, from the view of players that can "think truly and rationally" [1, 7, 10, 14, 15, 26, 27], it is best to use pure strategies from subsets (10). But how can their actions guarantee an appropriate result? Remember the maximin rule that is designed for such cases [1, 19]. Thus, using strategies from subsets

$$\hat{X}_{**} = \{x_{k_m^*}\}_{k_m^* \in K_{***}}$$

$$= \arg \max_{x_{k_m}, \, m=\overline{1, M_{**}}} \left\{ \min_{n=\overline{1, N_{**}}} \hat{a}_{mn} \right\} \subset \hat{X}_* \subset X_* \subset X \quad (14)$$

and

$$\hat{Y}_{**} = \{y_{l_n^*}\}_{l_n^* \in J_{***}}$$

$$= \arg \max_{y_{l_n}, \, n=\overline{1, N_{**}}} \left\{ \min_{m=\overline{1, M_{**}}} \hat{b}_{mn} \right\} \subset \hat{Y}_* \subset Y_* \subset Y \qquad (15)$$

guarantees for players their maximin payoffs

$$a_* = \max_{m=\overline{1, M_{**}}} \left\{ \min_{n=\overline{1, N_{**}}} \hat{a}_{mn} \right\} \qquad (16)$$

and

$$b_* = \max_{n=\overline{1, N_{**}}} \left\{ \min_{m=\overline{1, M_{**}}} \hat{b}_{mn} \right\} \qquad (17)$$

in any situation. Note that, however, not all situations in set $\hat{X}_{**} \times \hat{Y}_{**}$ are efficient Nash equilibria. Moreover, this set may not contain any Nash equilibria.

Now, make a denotation $L_{\bar{E}} = \hat{X}_{**} \times \hat{Y}_{**}$. If set

$$R_{\bar{E}} = L_{\bar{E}} \cap \hat{E} = \{\hat{X}_{**} \times \hat{Y}_{**}\} \cap \{\hat{z}_s\}_{s=1}^{S} \qquad (18)$$

is nonempty then it contains the refined Nash equilibria. In particular, if set (18) contains just a single element (an efficient Nash equilibrium), then the refinement is done. If $R_{\bar{E}} = \varnothing$ or $|R_{\bar{E}}| > 1$ then the superoptimality rule originally introduced for distinguishing optimal strategies in matrix games (see [28, 29]) can be applied.

If set $R_{\bar{E}} = \varnothing$ then using strategies from subsets (14) and (15) involves players into nonstability provoking them to search new pure strategies beyond these subsets for every game round (as there is no a single equilibrium point). In such a case, one of the best actions is to use strategies from subsets

$$\hat{X}_{***} = \{x_{k_m^{**}}\}_{k_m^{**} \in K_{***}}$$

$$= \arg \max_{x_{k_m}, \, m=\overline{1, M_{**}}} \left\{ \sum_{n=1}^{N_{**}} \hat{a}_{mn} \right\} \subset \hat{X}_* \subset X_* \subset X \quad (19)$$

and

$$\hat{Y}_{***} = \{y_{l_n^{**}}\}_{l_n^{**} \in J_{***}}$$

$$= \arg \max_{y_{l_n}, \, n=\overline{1, N_{**}}} \left\{ \sum_{m=1}^{M_{**}} \hat{b}_{mn} \right\} \subset \hat{Y}_* \subset Y_* \subset Y \qquad (20)$$

that guarantee for players their best payoffs under uncertainty of efficient equilibria in the bimatrix subgame with payoff matrices (13).

For the case of $|R_{\bar{E}}| > 1$ we need extra denotations. Let $R_{\bar{E}} \subset X_R \times Y_R$ by

$$\{x^{\langle R \rangle}, y^{\langle R \rangle}\} \in R_{\bar{E}}, \; x^{\langle R \rangle} \in X_R, \; y^{\langle R \rangle} \in Y_R,$$

and

$$M_R = |K_R| = |X_R|, \; N_R = |J_R| = |Y_R|,$$

with indices' sets $K_R = \{k_m\}_{m=1}^{M_R}$ and $J_R = \{j_n\}_{n=1}^{N_R}$. If $|X_R| > 1$ then

$$X_R^* = \{x_{k_m^*}\}_{k_m^* \in K_{R^*}} = \arg \max_{x_{k_m},\, m=\overline{1,\, M_R}} \left\{ \sum_{n=1}^{N_R} a_{k_m j_n} \right\}$$

$$\subset X_R \subset \hat{X}_{**} \subset \hat{X}_* \subset X_* \subset X . \qquad (21)$$

If $|Y_R| > 1$ then

$$Y_R^* = \{y_{l_n^*}\}_{l_n^* \in J_{R^*}} = \arg \max_{y_{l_n},\, n=\overline{1,\, N_R}} \left\{ \sum_{m=1}^{M_R} b_{k_m j_n} \right\}$$

$$\subset Y_R \subset \hat{Y}_{**} \subset \hat{Y}_* \subset Y_* \subset Y . \qquad (22)$$

Note that finding sets (21) and (22) does not guarantee that

$$\{X_R^* \times Y_R^*\} \cap R_{\bar{E}} \neq \varnothing . \qquad (23)$$

Statement (23) is only assuredly true for cases when either $|X_R| = 1$ or $|Y_R| = 1$.

An algorithm of using domination efficiency along with maximin and the superoptimality rule

Obviously, the algorithm cannot guarantee that a single efficient equilibrium will remain after the refinement [1, 2, 4, 8, 13, 15, 22, 24, 25, 30–34]. It has offshoots after 11 dual cases depending on what and how many elements sets E, (8), (10), (18), X_R, Y_R, (19), (20), (21), (22) consist of (Fig. 3). The algorithm has 10 definite steps at which the refinement is progressively accomplished.

1. It starts with actually checking whether the bimatrix game has equilibria or not.

2. If the game has a single Nash equilibrium then no refinement is needed.

3. If there are multiple Nash equilibria then set of efficient Nash equilibria is found.

4. No refinement is needed if a single efficient equilibrium exists.

5. For multiple efficient Nash equilibria, the refinement is done by maximizing the player's payoff when the other player has a single strategy corresponding to those efficient equilibria.

6. If both the players have multiple strategies corresponding to the efficient equilibria, a reduced bimatrix game is built on the product of those strategies.

7. Sets (14) and (15) are found (using the maximin rule).

8. If an intersection of the product of these sets and the efficient equilibria set is empty then sets (19) and (20) are found (using the superoptimality rule), whereupon refined equilibria are searched in an intersection of the product of sets (19) and (20) and the efficient equilibria set. If that intersection is empty then the algorithm only recommends for the players their best strategies by the superoptimality rule in sets (19) and (20), although these strategies do not constitute an efficient equilibrium. Otherwise, at least a refined efficient equilibrium is constituted.

9. If an intersection of the product of sets (14) and (15) and the efficient equilibria set is singleton then a single efficient Nash equilibrium is found. Otherwise, when this intersection is of multiple elements, sets (21) and (22) are found. Then the refinement is done by maximizing the player's payoff when the other player has a single strategy that produces set (18).

10. If set (18) is produced by the players' sets both consisting of multiple strategies, then at least a refined efficient equilibrium is found when an intersection of the product of sets (21) and (22) and set (18) is nonempty. If the intersection is empty then the algorithm only recommends for the players their best strategies by the superoptimality rule in sets (21) and (22), although these strategies do not constitute an efficient equilibrium.

There are two negative cases when the refinement fails: if conditions

$$\{\hat{X}_{***} \times \hat{Y}_{***}\} \cap \hat{E} = \varnothing \qquad (24)$$

and

$$\{X_R^* \times Y_R^*\} \cap R_{\bar{E}} = \varnothing \qquad (25)$$

turn true. Note that conditions (24) and (25) exclude each other. Moreover, even if case with (25) happens, it does not mean that the refinement fail is really total. Indeed, set (18) is not empty but it still contains "too many equilibria". Thus, owing to that $R_{\bar{E}} \subset \hat{E}$, such a case may be treated as a particular refinement — see an illustrative example in Fig. 4. In this example, holding at usual denotations of strategies,

$$R_{\bar{E}} = \{\{x_2, y_1\}, \{x_6, y_4\}, \{x_7, y_{10}\}\}$$

$$\subset \hat{E} = \{\{x_1, y_5\}, \{x_1, y_9\}, \{x_2, y_1\}, \{x_3, y_4\},$$

$$\{x_6, y_4\}, \{x_7, y_{10}\}\} \qquad (26)$$

that implies that a half of uncertainty in selecting the efficient equilibria is removed. Thus, those six

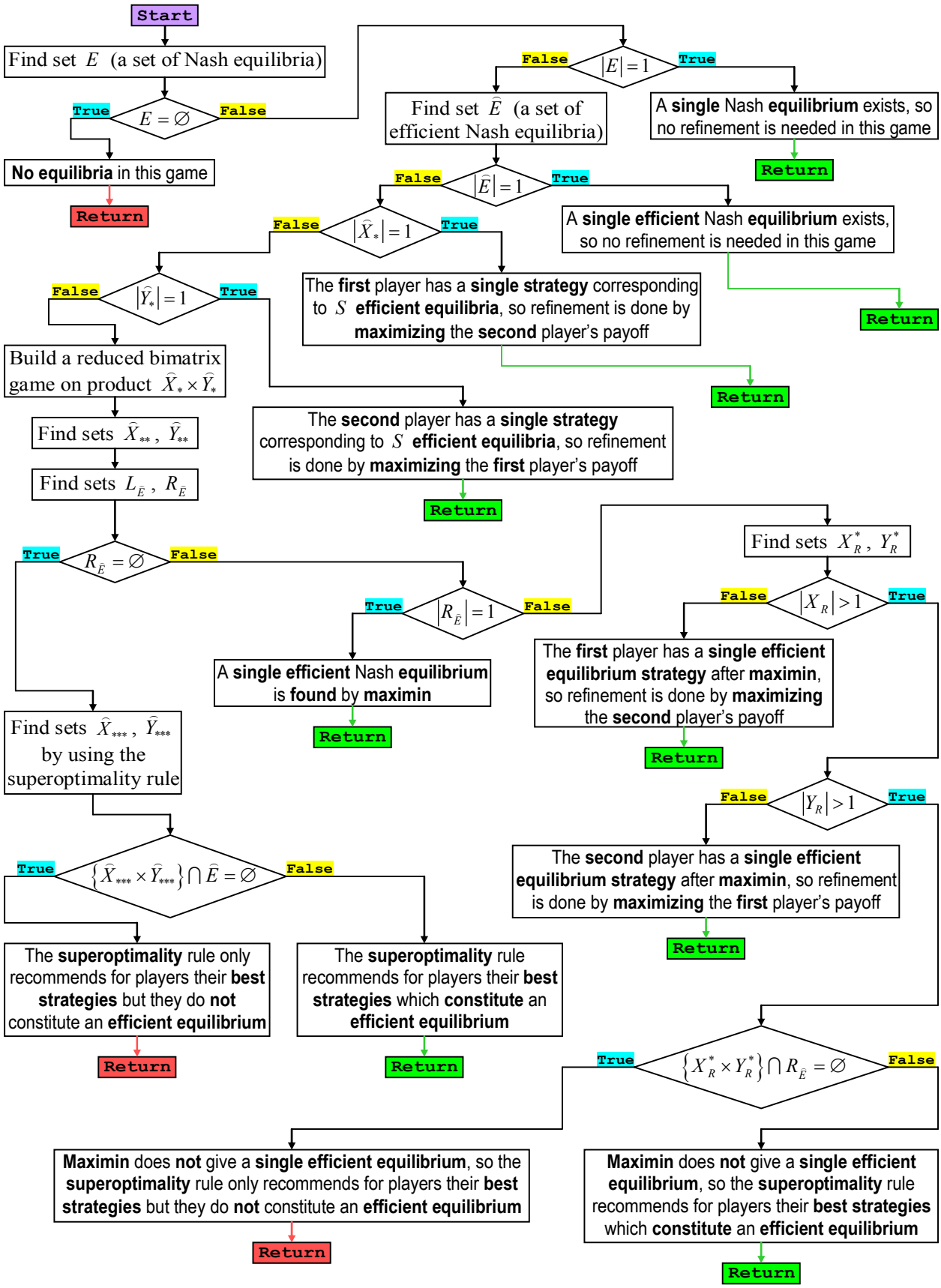

Fig. 3. The algorithm of the Nash equilibria refinement

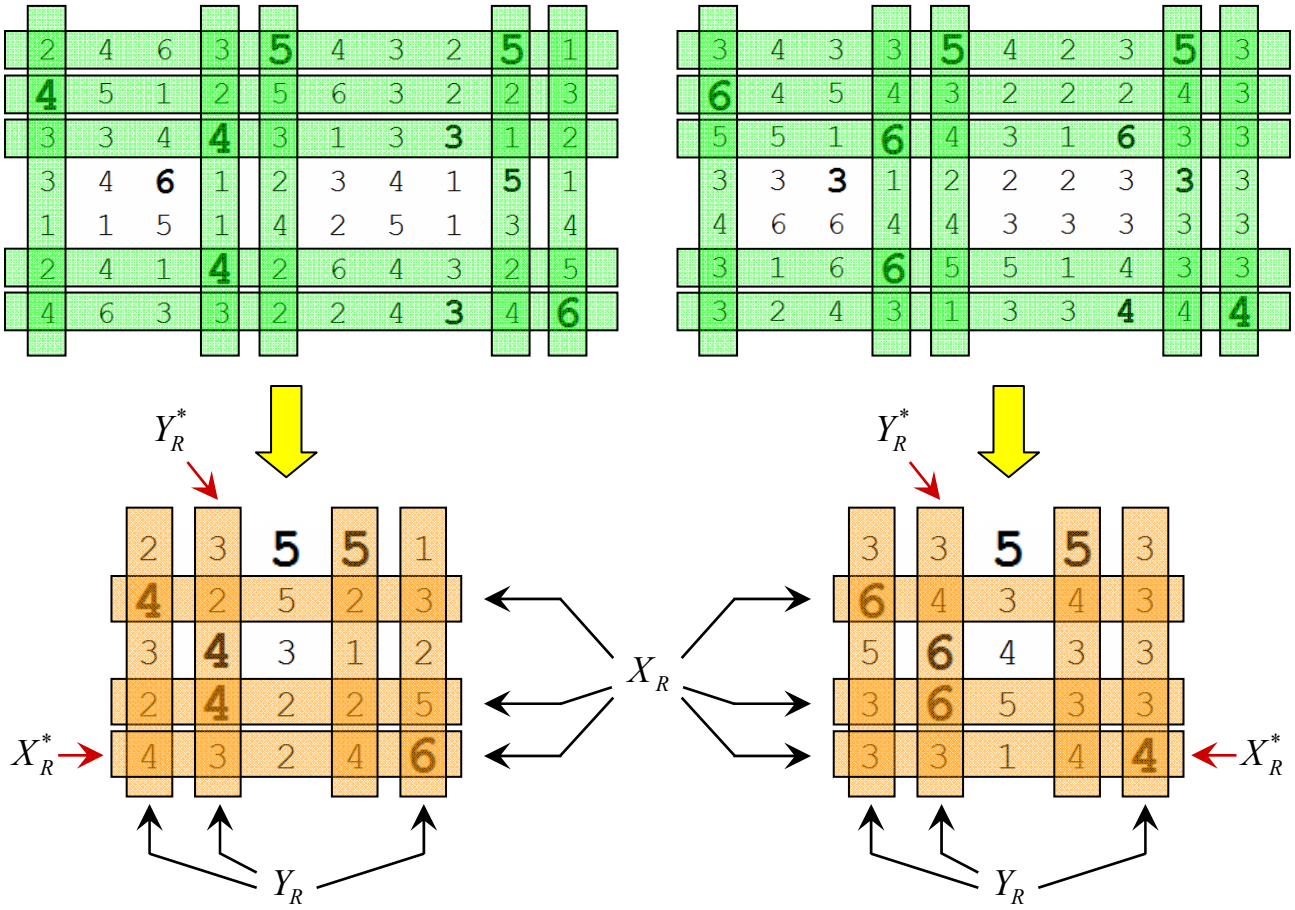

Fig. 4. Screenshots of payoff matrices of a bimatrix 7×10-game showing strategies of sets (10) as stripes corresponding to efficient equilibria highlighted bolder than the bold-highlighted non-efficient equilibria, and, after having reduced the game, strategies of maximin sets (14) and (15) as stripes; sets X_R and Y_R along with the resulting singletons (21) and (22) are arrowed

efficient equilibria are partially refined here [12, 15, 17, 31, 33, 34].

Figure 4 may seem to represent some paradoxical result wherein the resulting singletons (21) and (22) give a situation $\{x_7, y_4\}$ being even not an equilibrium at all. In this situation, players obtain equal but rather small payoffs $\{3, 3\}$. Might that be treated as a demerit of the developed algorithm? The answer is no, because nothing forces players to use just strategies from sets (21) and (22) when condition (23) fails. Amazingly enough, these strategies remain the "recommended best" under the uncertainty of the three situations in set $R_{\bar{E}} \subset \hat{E}$ – see its inclusion (26).

In another example, of the game with payoff matrices (3), where

$$\hat{\mathbf{A}} = (\hat{a}_{mn})_{3\times 4} = \begin{pmatrix} 5 & 2 & 4 & 2 \\ 5 & 5 & 3 & 1 \\ 2 & 6 & 3 & 6 \end{pmatrix} \text{ and}$$

$$\hat{\mathbf{B}} = (\hat{b}_{mn})_{3\times 4} = \begin{pmatrix} 3 & 4 & 6 & 4 \\ 4 & 2 & 3 & 4 \\ 3 & 3 & 0 & 3 \end{pmatrix}$$

are payoff matrices in the corresponding reduced game, a single efficient equilibrium is found by maximin. Here, $\hat{X}_{**} = \{x_1, x_3\}$ and $\hat{Y}_{**} = \{y_1, y_5\}$. Subsequently, $R_{\bar{E}} = \{x_3, y_5\}$ by the payoffs $\{6, 3\}$. This is an example of the totally successful refinement.

Discussion of advantages, disadvantages, and issues

It is clear that the algorithm works if one of the three following cases happens:

$$|R_{\bar{E}}| < |\hat{E}| \text{ by } R_{\bar{E}} \neq \varnothing, \qquad (27)$$

or intersection (23) holds, or

$$\{\hat{X}_{***} \times \hat{Y}_{***}\} \cap \hat{E} \neq \varnothing. \qquad (28)$$

An apparent advantage of the algorithm is in its giving a possibility to keep refining even if $R_{\hat{E}} = \varnothing$ (that is the maximin "fails"). Another advantage is in partially removing the uncertainty of equilibria by the superoptimality rule, without going into mixed strategies [20, 21, 23, 30, 35, 36]. Disadvantages are similar to those of the known refinement concepts – if payoffs come too "illogical" (illogically scattered) then the result may be some kind of frustration [1, 2, 4, 32, 34, 37, 38]. An example to this is

$$\mathbf{A} = (a_{mn})_{4\times 3} = \begin{pmatrix} 9 & 4 & 7 \\ 0 & 8 & 1 \\ 3 & 5 & 6 \\ 5 & 2 & 6 \end{pmatrix}$$

and

$$\mathbf{B} = (b_{mn})_{4\times 3} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 9 & 9 \\ 1 & 0 & 2 \\ 1 & 9 & 5 \end{pmatrix},$$

where

$$\hat{E} = \{\{x_1, y_1\}, \{x_2, y_2\}\}.$$

So, the game is reduced to matrices

$$\hat{\mathbf{A}} = (\hat{a}_{mn})_{2\times 2} = \begin{pmatrix} 9 & 4 \\ 0 & 8 \end{pmatrix} \text{ and } \hat{\mathbf{B}} = (\hat{b}_{mn})_{2\times 2} = \begin{pmatrix} 2 & 2 \\ 1 & 9 \end{pmatrix}$$

but maximin fails to give us an equilibrium – it gives just situation $\{x_1, y_2\}$ with such small payoffs $\{4, 2\}$. However, such an issue is resolved easily. The second player's second strategy in the reduced game does non-strictly dominate its first strategy, so the second player has no reason to use the first strategy ever. This is why only strategy y_2 will be clung to, and the first player understands that. Eventually, the first player has no reason to try get a payoff of 9 in situation $\{x_1, y_1\}$. Moreover, a payoff distance between that situation and situation $\{x_2, y_2\}$ is only 1 for the first player. Here the result is to cling to situation $\{x_2, y_2\}$ (both players should select their second strategies without doubts) and take payoffs $\{8, 9\}$.

An example of the game with payoff matrices

$$\mathbf{A} = (a_{mn})_{4\times 3} = \begin{pmatrix} 9 & 9 & 5 \\ 0 & 1 & 7 \\ 7 & 2 & 2 \\ 5 & 0 & 2 \end{pmatrix}$$

and

$$\mathbf{B} = (b_{mn})_{4\times 3} = \begin{pmatrix} 3 & 3 & 1 \\ 4 & 9 & 6 \\ 7 & 9 & 3 \\ 5 & 6 & 9 \end{pmatrix}$$

by

$$\hat{E} = \{\{x_1, y_1\}, \{x_1, y_2\}\}$$

goes to the case where the second player uses such strategy $y_{l^{**}} \in \hat{Y}_* = \{y_1, y_2\}$ at which its payoff is maximal. Both its payoffs are equal to 3, so the superoptimality rule prompts to use just y_2. Nevertheless, it is easy to see that actually strategy y_2 non-strictly dominates strategy y_1 making usage of y_1 senseless. The factual domination leaves us with a single equilibrium – situation $\{x_1, y_1\}$ and the corresponding payoffs $\{9, 3\}$. However, non-strictly dominated strategies are not always to be thrown away. For the illustrative example in Figure 4, strategy y_{10} is non-strictly dominated by strategy y_9. But situation $\{x_7, y_{10}\}$ is efficient and, if singleton Y_R^* had consisted of only strategy y_{10} instead of strategy y_4, this situation could have become a single refined equilibrium. This implies that throwing away dominated strategies must be done very carefully (not to be confused with throwing away strategies of dominated equilibria).

The developed concept of the equilibria refinement does not concern games with payoff symmetry and mirror-like symmetry [39, 40]. Dyadic games with payoff matrices (1), (2), and similar ones [1, 2, 4], as well as bimatrix games and finite noncooperative games of more players with identical/symmetric payoffs have Nash equilibria that are nonrefinable ever.

Conclusions

The suggested Nash equilibria refinement, based on the classical refinement by selecting only

efficient equilibria that dominate by their payoffs, exploits also maximin. The superoptimality rule is subsequently used mainly in two cases when a condition (24) or (25) turns true. In other words, the superoptimality rule is involved if maximin fails to produce just a single refined equilibrium. For a case when it produces more than a single refined equilibrium, the superoptimality rule may be used for a player having multiple refined equilibrium strategies but the other player has just a single refined equilibrium strategy. Here, nonetheless, simple selection of a maximal payoff may substitute the superoptimality rule if the payoffs of those multiple refined equilibrium strategies are different.

The contribution to the equilibria refinement game theory field can be advanced for cases when the developed algorithm gives more than one refined equilibrium [41, 42]. Besides, it can be attached to approximate Nash equilibrium situations with possible concessions [36, 43–45]. That all should be subsequently expanded into an equilibria refinement theory for finite noncooperative games of any number of players.

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В.В. Романюк

УДОСКОНАЛЕННЯ РІВНОВАГ НЕША В ЧИСТИХ СТРАТЕГІЯХ У БІМАТРИЧНИХ ІГРАХ ЗА ВИКОРИСТАННЯ ЕФЕКТИВНОСТІ ДОМІНУВАННЯ РАЗОМ ІЗ МАКСИМІНОМ І ПРАВИЛОМ НАДОПТИМАЛЬНОСТІ

Проблематика. Множинні рівноваги Неша породжують нову проблему вибору між ними, але ця проблема вирішується удосконаленням таких рівноваг. Однак жоден з існуючих методів удосконалення не може гарантувати єдину вдосконалену рівновагу Неша. У деяких іграх рівноваги Неша не підлягають удосконаленню.

Мета дослідження. Для вирішення проблеми неможливості вдосконалити рівноваги Неша в чистих стратегіях у біматричних іграх необхідно розробити алгоритм, який би якомога більше сприяв в удосконаленні рівноваг.

Методика реалізації. Пропонується метод удосконалення рівноваг Неша, заснований на класичному удосконаленні з виділенням тільки ефективних рівноваг, домінуючих за своїми виграшами. Згодом пропонований метод удосконалення використовує максимін. Правило надоптимальності залучається, якщо максимін не дає змоги отримати всього одну вдосконалену рівновагу.

Результати дослідження. Для удосконалення рівноваг Неша в біматричних іграх розроблено алгоритм із використанням ефективності домінування разом із максиміном і правилом надоптимальності. Алгоритм складається з 10 конкретних кроків, на яких поступово виконується удосконалення. Розроблена концепція удосконалення рівноваг не стосується ігор із симетрією або дзеркальною симетрією виграшів.

Висновки. Запропоноване удосконалення рівноваг Неша в чистих стратегіях є внеском в область удосконалення рівноваг у теорії ігор. Розроблений алгоритм дає змогу вибирати серед рівноваг Неша, що не підлягають удосконаленню, в біматричних іграх. Частково він усуває невизначеність рівноваг без переходу на змішані стратегії. Існують всього лише два негативних випадки, коли удосконалення не вдається. Для випадку, коли виробляється більше ніж одна удосконалена рівновага, правило надоптимальності може бути ужито гравцем, який має численні удосконалені рівноважні стратегії, а інший гравець має лише єдину удосконалену рівноважну стратегію.

Ключові слова: біматрична гра; рівноваги Неша; удосконалення; ефективність домінування; максимін; правило надоптимальності.

В.В. Романюк

УСОВЕРШЕНСТВОВАНИЕ РАВНОВЕСИЙ НЕША В ЧИСТЫХ СТРАТЕГИЯХ В БИМАТРИЧНЫХ ИГРАХ С ИСПОЛЬЗОВАНИЕМ ЭФФЕКТИВНОСТИ ДОМИНИРОВАНИЯ ВМЕСТЕ С МАКСИМИНОМ И ПРАВИЛОМ СВЕРХОПТИМАЛЬНОСТИ

Проблематика. Множественные равновесия Неша порождают новую проблему выбора между ними, но эта проблема решается усовершенствованием таких равновесий. Однако ни один из существующих методов усовершенствования не может гарантировать единственное усовершенствованное равновесие Неша. В некоторых играх равновесия Неша не подлежат усовершенствованию.

Цель исследования. Для решения проблемы невозможности усовершенствовать равновесия Неша в чистых стратегиях в биматричных играх необходимо разработать алгоритм, который бы как можно больше способствовал в усовершенствовании равновесий.

Методика реализации. Предлагается метод усовершенствования равновесий Неша, основанный на классическом усовершенствовании с выделением только эффективных равновесий, доминирующих по своим выигрышам. Впоследствии предлагаемый метод усовершенствования использует максимин. Правило сверхоптимальности привлекается, если максимин не позволяет получить всего одно усовершенствованное равновесие.

Результаты исследования. Для усовершенствования равновесий Неша в биматричных играх разработан алгоритм с использованием эффективности доминирования вместе с максимином и правилом сверхоптимальности. Алгоритм состоит из 10 конкретных шагов, на которых постепенно выполняется усовершенствование. Разработанная концепция усовершенствования равновесий не касается игр с симметрией или зеркальной симметрией выигрышей.

Выводы. Предложенное усовершенствование равновесий Неша в чистых стратегиях является вкладом в область усовершенствования равновесий в теории игр. Разработанный алгоритм позволяет выбирать среди не подлежащих усовершенствованию равновесий Неша в биматричных играх. Частично он устраняет неопределенность равновесий без перехода на смешанные стратегии. Существуют всего лишь два негативных случая, когда усовершенствование не удается. Для случая, когда производится более чем одно усовершенствованное равновесие, правило сверхоптимальности может быть применено игроком, имеющем множественные усовершенствованные равновесные стратегии, а другой игрок обладает лишь единственной усовершенствованной равновесной стратегией.

Ключевые слова: биматричная игра; равновесия Неша; усовершенствование; эффективность доминирования; максимин; правило сверхоптимальности.

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