ТЕОРЕТИЧНІ ТА ПРИКЛАДНІ ПРОБЛЕМИ МАТЕМАТИКИ

UDC 517.518.85+517.518.82
DOI: 10.20535/1810-0546.2017.4.108444

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BOUNDEDNESS OF LEBESGUE CONSTANTS AND INTERPOLATING FABER BASES[**]

Background. We investigate the relationship between the boundedness of Lebesgue constants for the Lagrange polynomial interpolation on a compact subset of $\mathbb{R}$ and the existence of a Faber basis in the space of continuous functions on this compact set.
Objective. The aim of the paper is to describe the conditions on the matrix of interpolation nodes under which the interpolation of any continuous function coincides with the decomposition of this function in a series on the Faber basis.
Methods. The methods of general theory of Schauder bases and the results which describe the convergence of interpolating Lagrange processes are used.
Results. The structure of matrices of interpolation nodes which generate the interpolating Faber bases is described.
Conclusions. Every interpolating Faber basis is generated by the interpolating Lagrange process with the interpolating matrix of a special kind and bounded Lebesgue constants.
Keywords: Lebesgue constant; Lebesgue function; Lagrange polynomial interpolation; Faber basis.

Introduction

The polynomial Lagrange interpolation is an important and widely used method for approximation of continuous functions. It is well known that if the domain of the function is massive enough, e.g. with nonnegative Lebesgue measure, then even under unlimited interpolation node increase the uniform convergence of the interpolation process can be guaranteed only for sufficiently smooth functions. For today, the divergence of Lagrange interpolation processes is studied in details. On the convergence of such processes, it is known much less. Namely, in the papers of S. Mergelyan [1] and P. Korovkin [2] the infinite compact subsets and the matrices of interpolation nodes which generate the uniform convergent sequences of interpolation polynomials for every continuous function were constructed. Moreover, the conditions of the uniform convergence of interpolating Lagrange polynomials for monotone sequences of interpolation nodes correlated to Faber bases of special kind (the interpolating Faber bases) were described by J. Obermaier and R. Szwarc (see, [3] and [4]). Note that J. Obermaier and R. Szwarc considered only the interpolation of function defined on compact subsets with unique limit points.

In this paper, we mainly study the uniform convergent Lagrange interpolation processes which correspond to arbitrary Faber bases.

For convenience, we recall the terminology which will be necessary for future.

Let

$$\mathfrak{M} = \{x_{k,n}\} = \begin{pmatrix} x_{1,1} & & & & \\ x_{1,2} & x_{2,2} & & & \\ \dots & \dots & \dots & & \\ x_{1,n} & x_{2,n} & \dots & x_{n,n} & \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

be an infinite triangular matrix which elements (nodes) are real numbers satisfying the condition $x_{k_1,n} \neq x_{k_2,n}$ for all distinct $k_1, k_2 \in \{1,\dots,n\}$ and every $n \in \mathbb{N}$. Then let's define the fundamental polynomials $l_{k_0,n} = l_{k_0,n}(\mathfrak{M},\cdot)$ as

$$l_{k_0,n}(x) = l_{k_0,n}(\mathfrak{M}, x)$$
$$:= \prod_{1\le k\le n, k\ne k_0} \frac{(x - x_{k,n})}{(x_{k_0,n} - x_{k,n})}, \quad x \in \mathbb{R}. \qquad (1)$$

The polynomials $l_{1,n},\dots,l_{n,n}$ form a basis at the linear space H_{n-1} of all real algebraic polynomials of degree at most $n-1$. In particular, we have $l_{1,1} \equiv 1$.

Let X be an infinite compact subset of $\mathbb{R}$. Let us denote by C_X the Banach space of continuous functions $f: X \to \mathbb{R}$ with the supremum norm

$$\| f \|_X := \sup\{| f(x)|: x \in X\}$$

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** The authors gratefully acknowledge the support of the project AMMODIT funded within EU H2020-MSCA-RICE

and write $\mathfrak{M} \subseteq X$ if $\mathfrak{M} = \{x_{k,n}\}$ and $x_{k,n} \in X$ for all $n \in \mathbb{N}$ and $k \le n$. For $f \in C_X$, $\mathfrak{M} \subseteq X$ and $n \in \mathbb{N}$, the Lagrange interpolating polynomial $L_n(f, \mathfrak{M}, \cdot)$ is the unique polynomial from H_n which coincides with f at the nodes $x_{k,n+1}$, $k = 1, \ldots, n+1$. Using the fundamental polynomials we can represent $L_n(f, \mathfrak{M}, \cdot)$ in the form

$$L_n(f, \mathfrak{M}, \cdot) = \sum_{k=1}^{n+1} f(x_{k,n+1}) l_{k,n+1}(\mathfrak{M}, \cdot). \tag{2}$$

For given X, $\mathfrak{M} \subseteq X$, and $n \in \mathbb{N}$, the Lebesgue function $\lambda_n(\mathfrak{M}, \cdot)$ and the Lebesgue constant $\Lambda_{n,X}(\mathfrak{M})$ can be defined as

$$\lambda_n(\mathfrak{M}, x) := \sup\{|L_n(f, \mathfrak{M}, x)| : \|f\|_X \le 1\}, \quad x \in \mathbb{R}, \tag{3}$$

and, respectively, as

$$\Lambda_{n,X}(\mathfrak{M}) := \sup\{\lambda_n(\mathfrak{M}, x) : x \in X\}. \tag{4}$$

The mappings

$$\mathfrak{L}_{n,\mathfrak{M}} : C_X \to C_X \quad \text{with} \quad \mathfrak{L}_{n,\mathfrak{M}}(f) = L_n(f, \mathfrak{M}, \cdot) \tag{5}$$

are bounded linear operators having the norms

$$\|\mathfrak{L}_{n,\mathfrak{M}}\| = \Lambda_{n,X}(\mathfrak{M}). \tag{6}$$

For every infinite compact set $X \subseteq \mathbb{R}$ and $\mathfrak{M} \subseteq X$ it is easy to prove that the equality

$$\lambda_n(\mathfrak{M}, x) = \sum_{k=1}^{n+1} |l_{k,n+1}(\mathfrak{M}, x)| \tag{7}$$

holds for each $x \in \mathbb{R}$.

Remark 1. Using formulas (1), (4) and (7), we can define the Lebesgue functions $\lambda_n(\mathfrak{M}, \cdot)$ and the Lebesgue constants $\Lambda_{n,X}(\mathfrak{M})$ for arbitrary nonempty set $X \subseteq \mathbb{R}$ and any interpolation matrix $\mathfrak{M} \subseteq \mathbb{R}$.

In what follows we will denote by ***BLC*** (bounded Lebesgue constants) the set of compact nonvoid sets $X \subseteq [-1, 1]$, for each of which there is a matrix $\mathfrak{M} \subseteq [-1, 1]$, such that the corresponding sequence $(\Lambda_{n,X}(\mathfrak{M}))_{n \in \mathbb{N}}$ is bounded, i.e.,

$$\Lambda_{n,X}(\mathfrak{M}) < c \tag{8}$$

holds for some $c > 0$ and every $n \in \mathbb{N}$.

Problem statement

The aim of the paper is to find the conditions under which the Lebesgue constants $\Lambda_{n,X}(\mathfrak{M})$ are bounded for the matrices $\mathfrak{M}$ having the form

$$\begin{pmatrix} x_{1,1} & & & & \\ x_{1,2} & x_{2,2} & & & \\ \ldots & \ldots & \ldots & & \\ x_{1,n} & x_{2,n} & \ldots & x_{n,n} & \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

Boundedness and convergence in Lagrange interpolation

J. Szabados and P. Vértesi, [5], write: "... in the convergence behavior of the Lagrange interpolatory polynomials ... the Lebesgue functions ... and the Lebesgue constants ... are of fundamental importance...".

Proposition 2. Let X be an infinite compact subset of $\mathbb{R}$ and let $\mathfrak{M} \subseteq X$. The following statements are equivalent.

(i) The inequality

$$\limsup_{n \in \mathbb{N}} \Lambda_{n,X}(\mathfrak{M}) < \infty$$

holds.

(ii) The limit relation

$$\lim_{n \to \infty} \|f - L_n(f, \mathfrak{M}, \cdot)\|_X = 0 \tag{9}$$

is valid for every $f \in C_X$.

(iii) The inequality

$$\limsup_{n \to \infty} \|L_n(f, \mathfrak{M}, \cdot)\|_X < \infty \tag{10}$$

holds for every $f \in C_X$.

Proof. The linear operator $\mathfrak{L}_{n,\mathfrak{M}}$ is a projection of C_X onto H_n. Hence, by Lebesgue's lemma, see [6, Ch. 2, Pr. 4.1], we have the inequality

$$\|L_n(f, \mathfrak{M}, \cdot) - f\|_X \le (1 + \Lambda_{n,X}(\mathfrak{M})) E_n(f) \tag{11}$$

where $E_n(f)$ is the error of the best approximation of f by H_n in C_X. By the Stone–Weierstrass theorem, the continuous function f is uniformly approximable by polynomials on X, i.e., $\lim_{n \to \infty} E_n(f) = 0$. Now $(i) \Rightarrow (ii)$ follows.

The implication $(ii) \Rightarrow (iii)$ is trivial.

Suppose that (iii) holds. To prove $(iii) \Rightarrow (i)$ note that equality (10) implies the boundedness of sequences

$$(\|\mathfrak{L}_{n,\mathfrak{M}}(f)\|_X)_{n \in \mathbb{N}} = (\|L_n(f, \mathfrak{M}, \cdot)\|_X)_{n \in \mathbb{N}}$$

for every $f \in C_X$. Since all $\mathfrak{L}_{n,\mathfrak{M}} : C_X \to C_X$ are continuous linear operators and C_X is a Banach space, the Banach–Steinhaus theorem gives us the inequality

$$\sup_{n\in\mathbb{N}} \|\mathfrak{L}_{n,\mathfrak{M}}\| < \infty.$$

The last inequality and (6) imply (i). □

There is a pointwise analog of Proposition 2.

Proposition 3. Let X be an infinite compact subset of $\mathbb{R}$ and let $x \in X$. The following statements are equivalent for every $\mathfrak{M} \subseteq X$.

(i) The inequality

$$\limsup_{n\to\infty} \lambda_n(\mathfrak{M}, x) < \infty \tag{12}$$

holds.

(ii) The limit relation

$$\lim_{n\to\infty} L_n(f, \mathfrak{M}, x) = f(x)$$

is valid for every $f \in C_X$.

(iii) The inequality

$$\limsup_{n\to\infty} |L_n(f, \mathfrak{M}, x)| < \infty \tag{13}$$

holds for every $f \in C_X$.

Proof. Using (3) instead of (6) and the inequality

$$|f(x) - L_n(f, \mathfrak{M}, x)| \leq (1 + \lambda_n(\mathfrak{M}, x))E_n(f)$$

(see [5, p. 6]) instead of (11), we can prove (i) ⇒ (ii) as in the proof of Proposition 2. The implication (ii) ⇒ (iii) is trivial. The Banach–Steinhaus theorem and (3) give us the implication (iii) ⇒ (i). □

Corollary 4. Let X be an infinite compact subset of $\mathbb{R}$ and let $\mathfrak{M} \subseteq X$. The sequence $(\lambda_n(\mathfrak{M}, \cdot))_{n\in\mathbb{N}}$ is pointwise bounded on X if and only if the sequence $(L_n(f, \mathfrak{M}, \cdot))_{n\in\mathbb{N}}$ is pointwise convergent to f on X for every $f \in C_X$.

For the classical case $X = [-1,1]$ there exist a lot of important results connected with the unboundedness of the Lebesgue constants and the Lebesgue functions.

In 1914, G. Faber [7], for every matrix $\mathfrak{M} \subseteq [-1,1]$, proved the existence of $f \in C_{[-1,1]}$ satisfying the inequality

$$\limsup_{n\to\infty} \|f - L_n(f, \mathfrak{M}, \cdot)\|_{[-1,1]} > 0 \tag{14}$$

that, by Proposition 2, is an equivalent for

$$\limsup_{n\to\infty} \Lambda_{n,[-1,1]}(\mathfrak{M}) = \infty. \tag{15}$$

In 1931, S.N. Bernstein [8] found that for every $\mathfrak{M} \subseteq [-1,1]$ there are $f \in C_{[-1,1]}$ and $x_0 \in [-1,1]$ such that

$$\limsup_{n\to\infty} |L_n(f, \mathfrak{M}, x_0)| = \infty. \tag{16}$$

This equality together with Proposition 3 gives the existence of a point $x_0 \in [-1,1]$ satisfying

$$\limsup_{n\to\infty} \lambda_n(\mathfrak{M}, x_0) = \infty. \tag{17}$$

In 1980, P. Erdös and P. Vértesi [9] proved the following theorem.

Theorem 5. Let $\mathfrak{M} \subseteq [-1,1]$. Then there is $f \in C_{[-1,1]}$ such that limit relation (16) holds for almost all $x_0 \in [-1,1]$.

This theorem implies the following corollary.

Corollary 6. Let X be an infinite compact subset of $\mathbb{R}$. Let us denote by $m_1(X)$ the one-dimensional Lebesgue measure of X. Write

$$a = \min\{x : x \in X\} \quad \text{and} \quad b = \max\{x : x \in X\}.$$

If there is $\mathfrak{M} \subseteq [a,b]$ such that inequality (12) holds for every $x \in X$, then X is nowhere dense and

$$m_1(X) = 0. \tag{18}$$

Proof. Since the fundamental polynomials are invariant under the affine transformations of $\mathbb{R}$, we may suppose that $a = -1$ and $b = +1$. Now, (18) follows from Theorem 5. Equality (18) implies that the interior of X is empty, $\operatorname{Int} X = \varnothing$. Since X is compact, we have $\bar{X} = X$, where $\bar{X}$ is the closure of X. Consequently, the equality $\operatorname{Int} \bar{X} = \varnothing$ holds, it means that X is nowhere dense. □

Corollary 7. If X belongs to ***BLC***, then X is nowhere dense in $\mathbb{R}$ and its one-dimensional Lebesgue measure is zero.

Example 8. If $X = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots\}$ is a dense subset of $[-1,1]$ and the matrix $\mathfrak{M}$ is defined such that $x_{k,n} = x_k$ for all $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$, then we evidently have the equalities

$$\limsup_{n\to\infty} \lambda_n(\mathfrak{M}, x) = \lim_{n\to\infty} \lambda_n(\mathfrak{M}, x) = 1 \tag{19}$$

for every $x \in X$. Consequently, the compactness of X cannot be dropped in Corollary 6.

It was proved by A.A. Privalov in [10], that there are a countable set $X \subseteq [0,1]$ and a positive constant $c_1 = c_1(X)$, such that 0 is the unique accumulation point of X and the inequality

$$\Lambda_{n,X}(\mathfrak{M}) \ge c_1 \ln(n+1)$$

holds for every $n \in \mathbb{N}$ and every $\mathfrak{M} \subseteq [-1,1]$.

Remark 9. There is a constant $c_2 > 0$ for which

$$\Lambda_{n,[-1,1]}(\mathfrak{M}) \le c_2 \ln(n+1)$$

holds for every $n \in \mathbb{N}$ with $\mathfrak{M} = \{x_{k,n}\}$ based on the Chebyshev nodes $x_{k,n} = \cos\frac{(2k-1)\pi}{2n}$. For details see [11].

The example of perfect set $X \in \boldsymbol{BLC}$ was obtained by S.N. Mergelyan [1].

P.P. Korovkin [2] found a perfect $X \subseteq [-1,1]$ and a matrix $\mathfrak{M}$ such that, for every $f \in C_X$, the sequence $(L_{n^2}(f, \mathfrak{M}, \cdot))_{n\in\mathbb{N}}$ uniformly tends to f,

$$\sup_{n\in\mathbb{N}} \Lambda_{n^2,X}(\mathfrak{M}) < \infty.$$

At the same paper [2], he wrote that there is a modification of X with bounded sequence of Lebesgue constants.

Corollary 7 indicates that every $X \in \boldsymbol{BLC}$ must be small in a very strong sense. Moreover, the examples of A.A. Privalov, P.P. Korovkin, and S.N. Mergelyan show that the properties “are countable” and “belong to the class ***BLC***” not linked too closely.

Faber bases and Lagrange polynomials

In what follows we study the boundedness of the Lebesgue constants $\Lambda_{n,X}(\mathfrak{M})$ for the matrices $\mathfrak{M}$ having the form

$$\begin{pmatrix} x_1 & & & & \\ x_1 & x_2 & & & \\ \dots & \dots & \dots & & \\ x_1 & x_2 & \dots & x_n & \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

The obtained results are inspired by some ideas of J. Obermaier and R. Szwarc [3, 4].

Let X be an infinite compact subset of $\mathbb{R}$.

Definition 10. A Faber basis in C_X is a sequence $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ of real algebraic polynomials satisfying the following conditions:

(i) for every $f \in C_X$ there is a unique sequence $(a_k)_{k\in\mathbb{N}}$ of real numbers such that

$$f = \sum_{k=1}^{\infty} a_k p_k; \qquad (20)$$

(ii) for every $k \in \mathbb{N}$ the polynomial p_k has the degree $k-1$, $\deg p_k = k-1$.

Remark 11. As usual, equality (20) means that

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} a_k p_k \right\|_X = 0.$$

Let $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ be a Faber basis in C_X. For every $f \in C_X$ we shall denote by $S_{n,\tilde{p}}(f)$ the partial sum $\sum_{k=1}^{n} a_k p_k$ of series (20), i.e.,

$$S_{n,\tilde{p}}(f) = \sum_{k=1}^{n} a_k p_k.$$

If $n \in \mathbb{N}$ is given, then the partial sum operator $S_{n,\tilde{p}}: C_X \to C_X$ is a linear operator with the range H_{n-1} and the domain C_X. Similarly, for an interpolation matrix $\mathfrak{M} \subseteq X$, the operator, defined by (5),

$$\mathfrak{L}_{n,\mathfrak{M}}: C_X \to C_X,$$

has the same range and domain. Moreover, the linear operators $S_{n,\tilde{p}}$ and $\mathfrak{L}_{n,\mathfrak{M}}$ are projections on H_{n-1}, i.e., we have

$$S_{n,\tilde{p}}(p) = \mathfrak{L}_{n,\mathfrak{M}}(p) = p$$

for every $p \in H_{n-1}$. In what follows we study some conditions under which the operators $S_{n,\tilde{p}}$ and $\mathfrak{L}_{n,\mathfrak{M}}$ are the same for every $n \in \mathbb{N}$.

Definition 12. A Faber basis $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ is interpolating if there is a sequence $(x_k)_{k\in\mathbb{N}}$ of distinct points of X such that the equality

$$S_{k,\tilde{p}}(f)(x_k) = f(x_k) \qquad (21)$$

holds for all $f \in C_X$ and $k \in \mathbb{N}$.

If $\tilde{p}$ and $(x_k)_{k\in\mathbb{N}}$ satisfy the above condition, then we say that $\tilde{p}$ is interpolating with the nodes $(x_k)_{k\in\mathbb{N}}$.

Remark 13. The interpolating Faber bases are a particular case of the interpolating Schauder bases for a space of continuous functions on a locally compact metric space, [12, Definition 1.3.1].

The following lemma is similar to Proposition 1.3.2 from [12].

Lemma 14. Let X be an infinite compact subset of $\mathbb{R}$, let $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ be a Faber basis in C_X and let $(x_k)_{k\in\mathbb{N}}$ be a sequence of distinct points of X. Then $\tilde{p}$ is interpolating with the nodes $(x_k)_{k\in\mathbb{N}}$ if and only if

$$p_k(x_k) \neq 0 \quad \text{and} \quad p_k(x_j) = 0 \tag{22}$$

for every $k \in \mathbb{N}$ and $j < k$.

P r o o f. Suppose that $\tilde{p}$ is interpolating with the nodes $(x_k)_{k\in\mathbb{N}}$. We must show that (22) holds for all $k \in \mathbb{N}$ and $j < k$. Since, for each $f \in C_X$, the representation

$$f = \sum_{k=1}^{\infty} a_k p_k \tag{23}$$

is unique, we have

$$p_k \neq 0 \tag{24}$$

for every $k \in \mathbb{N}$. The equality $\deg p_1 = 0$ together with (24) implies (22) for $k = 1$. Let $k \geq 2$. The uniqueness of representation (23) gives us the equalities

$$S_{1,\tilde{p}}(p_k) = \ldots = S_{k-1,\tilde{p}}(p_k) = 0. \tag{25}$$

Since $\tilde{p}$ is interpolating with the nodes $(x_k)_{k\in\mathbb{N}}$, (25) implies

$$p_k(x_1) = \ldots = p_k(x_{k-1}) = 0.$$

If $p_k(x_k) = 0$, then p_k has k distinct zeros that contradicts the equality $\deg p_k = k-1$. Condition (22) follows.

Let (22) hold for all $k \in \mathbb{N}$ and $j < k$. Then from (23) we obtain

$$f(x_n) = \sum_{k=1}^{\infty} a_k p(x_n) = \sum_{k=1}^{n} a_k p_k(x_n) = S_{n,\tilde{p}}(f)(x_n)$$

for every $n \in \mathbb{N}$. Thus, $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ is interpolating with the nodes $(x_k)_{k\in\mathbb{N}}$. □

Corollary 15. Let X be an infinite compact subset of $\mathbb{R}$ and let $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ be an interpolating Faber basis in C_X. Then there is a unique sequence $(x_k)_{k\in\mathbb{N}}$. of distinct points of X such that $\tilde{p}$ is interpolating with nodes $(x_k)_{k\in\mathbb{N}}$.

P r o o f. Let $\tilde{p}$ be interpolating with nodes $(x_k)_{k\in\mathbb{N}}$. By Lemma 14 the point x_1 is the unique zero of the polynomial p_2, the point x_2 can be characterized as the unique point of X for which $p_3(x_2) = 0$ and $p_2(x_2) \neq 0$ and so on. □

Lemma 14 implies also the following

Proposition 16. Let X be an infinite compact subset of $\mathbb{R}$. If $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ is an interpolating Faber basis in C_X with nodes $(x_k)_{k\in\mathbb{N}}$, then for every sequence $\tilde{\lambda} = (\lambda_k)_{k\in\mathbb{N}}$ of nonzero real numbers the sequence

$$\tilde{\lambda}\tilde{p} = (\lambda_k p_k)_{k\in\mathbb{N}}$$

is also an interpolating Faber basis with the same nodes $(x_k)_{k\in\mathbb{N}}$. Conversely, if $\tilde{q} = (q_k)_{k\in\mathbb{N}}$ and $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ are interpolating Faber bases with the same nodes, then there is a unique sequence $\tilde{\mu} = (\mu_k)_{k\in\mathbb{N}}$ of nonzero real numbers such that

$$\tilde{q} = \tilde{\mu}\tilde{p} = (\mu_k p_k)_{k\in\mathbb{N}}.$$

For given nodes $(x_k)_{k\in\mathbb{N}}$, the interpolating Faber basis $\tilde{p} = (p_k)_{k\in\mathbb{N}}$, if such a basis exists, can be uniquely determined by the natural normalization

$$p_k(x_k) = 1$$

for every $k \in \mathbb{N}$.

Definition 17 [3]. A Faber basis $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ is called a Lagrange basis with respect to the sequence $(x_k)_{k\in\mathbb{N}}$ if

$$p_k(x_k) = 1 \quad \text{and} \quad p_k(x_j) = 0 \tag{26}$$

for all $k \in \mathbb{N}$ and $j < k$.

The following example gives us another condition of interpolating Faber basis uniqueness corresponding to given nodes. Recall that a polynomial is *monic* if its leading coefficient is equal to 1.

Example 18. Let $\tilde{\pi} = (\pi_k)_{k\in\mathbb{N}}$ be an interpolating Faber basis with nodes $(x_k)_{k\in\mathbb{N}}$ and monic polynomials π_k. Then $\pi_1, \pi_2, \ldots, \pi_k, \ldots$ are the Newton polynomials,

$$\pi_k(x) = \begin{cases} 1 & \text{if} \quad k = 1, \\ \prod_{j=1}^{k-1}(x - x_j) & \text{if} \quad k \geq 2. \end{cases} \tag{27}$$

The sequence $\tilde{p} = (p_k)_{k\in\mathbb{N}}$,

$$p_k = \begin{cases} 1 & \text{if} \quad k = 1, \\ \dfrac{\pi_k}{\prod_{j=1}^{k-1}(x_k - x_j)} & \text{if} \quad k \geq 2, \end{cases} \tag{28}$$

is a Lagrange basis with respect to $(x_k)_{k\in\mathbb{N}}$.

Theorem 19. Let X be an infinite compact subset of $\mathbb{R}$ and let $(x_k)_{k\in\mathbb{N}}$ be a sequence of distinct points of X. The following two statements are equivalent.

(i) There is an interpolating Faber basis with the nodes $(x_k)_{k\in\mathbb{N}}$.

(ii) For every $f \in C_X$ we have

$$f = \sum_{k=1}^{\infty} f[x_1,...,x_k]\pi_k \tag{29}$$

where, for each $k \in \mathbb{N}$, π_k is the Newton polynomials defined by (27) and $f[x_1,...,x_k]$ is the divided difference of the function f,

$$f[x_1] = f(x_1),\ f[x_1, x_2] = \frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_2 - x_1}, \ldots,$$

$$f[x_1,...,x_k] = \sum_{j=1}^{k} \frac{f(x_j)}{\prod_{i=1, i\neq j}^{k}(x_j - x_i)}.$$

P r o o f. (i) $\Rightarrow$ (ii). If (i) holds, then by Lemma 14, $\tilde{\pi} = (\pi_k)_{k\in\mathbb{N}}$ is an interpolating Faber basis in C_X with nodes $(x_k)_{k\in\mathbb{N}}$. Consequently, for every $f \in C_X$ there is a unique sequence $(y_k)_{k\in\mathbb{N}}$ such that

$$f = \sum_{k=1}^{\infty} y_k \pi_k. \tag{30}$$

Since the basis $\tilde{\pi}$ is interpolating, we have

$$\begin{cases} y_1\pi_1(x_1) = f(x_1), \\ y_1\pi_1(x_2) + y_2\pi_2(x_2) = f(x_2), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_1\pi_1(x_k) + y_2\pi_2(x_k) + ... + y_k\pi_k(x_k) = f(x_k). \end{cases} \tag{31}$$

The polynomial

$$f[x_1]\pi_1 + ... + f[x_1,...,x_k]\pi_k$$

coincides with the function f at the points $x_1,...,x_k$. (See Theorem 1.1.1 and formula (1.19) in [13] for details). Since linear system (31) has a unique solution, we have

$$y_1 = f[x_1],..., y_k = f[x_1,...,x_k]. \tag{32}$$

Equality (29) follows.

(ii) $\Rightarrow$ (i). Let (ii) hold. Then, the sequence $\tilde{\pi} = (\pi_k)_{k\in\mathbb{N}}$ is an interpolating Faber basis in C_X if and only if (30) implies (32) for every $f \in C_X$ and every $k \in \mathbb{N}$, that follows from the uniqueness of solutions of (31). □

Theorem 20. Let X be an infinite compact subset of $\mathbb{R}$ and let $\mathfrak{M} = \{x_{k,n}\}$ be an interpolation matrix with the nodes in X. The following conditions are equivalent.

(i) The space C_X admits a Faber basis $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ such that the equality

$$S_{n,\tilde{p}} = \mathfrak{L}_{n,\mathfrak{M}} \tag{33}$$

holds for every $n \in \mathbb{N}$.

(ii) The sequence $(\Lambda_{n,X}(\mathfrak{M}))_{n\in\mathbb{N}}$ is bounded and there is a sequence $(x_k)_{k\in\mathbb{N}}$ of distinct points of X such that for any $n \geq 2$ the tuple $(x_{1,n},...,x_{n,n})$ is a permutation of the set $\{x_1,...,x_n\}$.

P r o o f. (i) $\Rightarrow$ (ii). Let $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ be a Faber basis in C_X and let (33) hold for every $n \in \mathbb{N}$. The partial sum operators are bounded for every Faber basis. (See, for example, [12, Proposition 1.1.4]). Hence, we have

$$\sup_n \|S_{n,\tilde{p}}\| < \infty.$$

The last inequality and (33) imply the boundedness of the sequence $(\Lambda_{n,X}(\mathfrak{M}))_{n\in\mathbb{N}}$. Now to prove (ii) it suffices to show that for every $n \geq 2$ and every $k_1 \leq n$ there is $k_2 \leq n+1$ such that

$$x_{k_1,n} = x_{k_2,n+1}$$

holds. Suppose that, on the contrary, there is $n \geq 2$ and $k_1 \in \{1,..,n\}$ such that

$$x_{k_1,n} \neq x_{k_2,n+1}$$

for all integer numbers $k_2 \in \{1,...,n+1\}$. We can find a function $f \in C_X$ satisfying the equalities

$$f(x_{k_1,n}) = 1$$

and

$$f(x_{1,n+1}) = f(x_{2,n+1}) = ... = f(x_{n+1,n+1}) = 0.$$

These equalities imply that

$$\mathfrak{L}_{n+1,\mathfrak{M}}(f) = L_n(f, \mathfrak{M}, \cdot) = 0$$

and

$$\mathfrak{L}_{n,\mathfrak{M}}(f) = L_{n-1}(f, \mathfrak{M}, \cdot) \neq 0.$$

Now, using the obvious equality

$$S_{n,\tilde{p}} \circ S_{n+1,\tilde{p}} = S_{n,\tilde{p}}$$

and (33) we obtain the contradiction

$$0 \neq \mathfrak{L}_{n,\mathfrak{M}}(f) = S_{n,\tilde{p}}(f)$$

$$= S_{n,\tilde{p}}(S_{n+1,\tilde{p}}(f)) = S_{n,\tilde{p}}(\mathfrak{L}_{n+1,\mathfrak{M}}(f)) = S_{n,\tilde{p}}(0) = 0.$$

Statement (ii) follows.

(ii) $\Rightarrow$ (i). Let (ii) hold. The boundedness of the sequence $(\Lambda_{n,X}(\mathfrak{M}))_{n\in\mathbb{N}}$ implies that

$$\lim_{n\to\infty} \| f - L_n(f, \mathfrak{M}, \cdot)\|_X = 0 \tag{34}$$

holds for every $f \in C_X$. (See Proposition 2). Since the Lagrange interpolation polynomial $L_n(f, \mathfrak{M}, \cdot)$ is invariant with respect to arbitrary permutation of the nodes $x_{1,n+1}, \ldots, x_{n+1,n+1}$, we may suppose that

$$x_{1,n+1} = x_1, , x_{2,n+1} = x_2, \ldots, x_{n+1,n+1} = x_{n+1}$$

for every $n \in \mathbb{N}$. Using the Newton polynomials π_k (see (27)) we may write the polynomial $L_n(f, \mathfrak{M}, \cdot)$ in the form

$$L_n(f, \mathfrak{M}, \cdot) = f[x_1]\pi_1 + f[x_1, x_2]\pi_2$$

$$+ \ldots + f[x_1, x_2, \ldots, x_{n+1}]\pi_{n+1}. \tag{35}$$

Hence, we have the representation

$$f = \sum_{k=1}^{\infty} f[x_1, \ldots, x_k]\pi_k.$$

Now, (i) follows from Theorem 19. □

Corollary 21. Let X be an infinite compact subset of $\mathbb{R}$ and let $\mathfrak{M} \subseteq X$ be an interpolation matrix with bounded $(\Lambda_{n,X}(\mathfrak{M}))_{n\in\mathbb{N}}$. Then the following conditions are equivalent.

(i) There is a Faber basis of C_X such that (33) holds for every $n \in \mathbb{N}$.

(ii) The equality

$$\mathfrak{L}_{n,\mathfrak{M}} \circ \mathfrak{L}_{n+1,\mathfrak{M}} = \mathfrak{L}_{n+1,\mathfrak{M}} \circ \mathfrak{L}_{n,\mathfrak{M}}$$

holds for every $n \in \mathbb{N}$.

(iii) The inequality

$$\deg L_n(f, \mathfrak{M}, \cdot) \geq \deg L_{n-1}(f, \mathfrak{M}, \cdot)$$

holds for every $n \in \mathbb{N}$ and every $f \in C_X$.

P r o o f. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow directly from Definition 10. The proofs of (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) are similar to the proof (i) $\Rightarrow$ (ii) in Theorem 20. □

Remark 22. Statements (ii) and (iii) of Corollary 21 can be considered as some special cases of Lemma 4.7 in [14] and Theorem 20.1 in [15] respectively.

Lemma 23. Let X be an infinite compact subset of $\mathbb{R}$. The following statements are equivalent for arbitrary Faber bases $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ and $\tilde{q} = (q_k)_{k\in\mathbb{N}}$ in C_X.

(i) There is a sequence $\tilde{\lambda} = (\lambda_k)_{k\in\mathbb{N}}$ of nonzero numbers such that

$$p_k = \lambda_k q_k \tag{36}$$

holds for every $k \in \mathbb{N}$.

(ii) The equality

$$S_{n,\tilde{p}} = S_{n,\tilde{q}} \tag{37}$$

holds for every $n \in \mathbb{N}$.

P r o o f. The implication (i) $\Rightarrow$ (ii) follows from the definition of the Faber bases in C_X.

Let (ii) hold. Equality (36) is trivial if $k = 1$. If $k \geq 2$, then there are $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$p_k = \sum_{i=1}^{k} \lambda_i q_i = \lambda_k q_k + S_{k-1,\tilde{q}}(p_k).$$

Using (37) we obtain

$$S_{k-1,\tilde{q}}(p_k) = S_{k-1,\tilde{p}}(p_k) = 0.$$

Hence $p_k = \lambda_k q_k$ holds. Moreover, we have $\lambda_k \neq 0$ because

$$\deg p_k = \deg q_k = k - 1. \; \square$$

The following theorem is a dual form of Theorem 20 and it can be considered as the main result of this section of the paper.

Theorem 24. Let X be an infinite compact subset of $\mathbb{R}$ and let $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ be a Faber basis in C_X. The following conditions are equivalent.

(i) There exists an interpolation matrix $\mathfrak{M} \subseteq X$ such that equality (33) holds for every $n \in \mathbb{N}$.

(ii) The basis $\tilde{p}$ is interpolating.

P r o o f. (i) $\Rightarrow$ (ii). Let $\mathfrak{M} = \{x_{n,k}\}$ be an interpolation matrix such that $\mathfrak{M} \subseteq X$ and the equality

$$\mathfrak{L}_{n,\mathfrak{M}} = S_{n,\tilde{p}}$$

holds for every $n \in X$. Using Theorem 20 we can suppose that there is a sequence $(x_k)_{k\in\mathbb{N}}$ of distinct points of X such that

$$x_{k,n} = x_k$$

for all $n \ge 1$ and $k \in \{1,\dots,n\}$. To prove (ii) it suffices to show that $\tilde{p}$ is interpolating with nodes $(x_k)_{k\in\mathbb{N}}$. As in the proof of implication (ii) $\Rightarrow$ (i) from Theorem 20 we obtain that the basis $\tilde{\pi} = (\pi_k)_{k\in\mathbb{N}}$ consisting of the corresponding Newton polynomials is an interpolating Faber basis with the nodes $(x_k)_{k\in\mathbb{N}}$ for which the equality

$$\mathfrak{L}_{n,\mathfrak{M}} = S_{n,\tilde{\pi}} \tag{39}$$

holds for every $n \in \mathbb{N}$. (See equality (35).) By Lemma 23, it follows from (38) and (39) that there is a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of nonzero real numbers such that

$$p_k = \lambda_k \pi_k$$

holds for every $k \in \mathbb{N}$. Since $\tilde{\pi}$ is an interpolating Faber basis with nodes $(x_k)_{k\in\mathbb{N}}$, Proposition 16 implies that $\tilde{p}$ is also interpolating with the same nodes.

(ii) $\Rightarrow$ (i). Suppose that $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ is interpolating with nodes $(x_k)_{k\in\mathbb{N}}$. If

$$\tilde{p} = \tilde{\pi},$$

where $\tilde{\pi} = (\pi_k)_{k\in\mathbb{N}}$ is the interpolating basis consisting of the Newton polynomials, then using Theorem 19 we can show that (33) holds for all $n \in \mathbb{N}$ with

$$\mathfrak{M} = \{x_{k,n}\}, \quad x_{k,n} = x_k, \quad n \in \mathbb{N}, \quad k \in \{1,\dots,n\}.$$

The case of an arbitrary interpolating Faber basis $\tilde{p} = (p_k)_{k\in\mathbb{N}}$ can be reduced to the case $\tilde{p} = \tilde{\pi}$ with the help of Lemma 23 and Proposition 16. □

Conclusions

Some new details of the well-known interplay between the boundedness of Lebesgue constants and the uniform convergence of Lagrange polynomials are described. The corresponding relationships of pointwise boundedness of Lebesgue functions with pointwise convergence of these polynomials are described as well. Some new relations between the boundedness of Lebesgue constants for special interpolating matrices and the existence of interpolating Faber bases are obtained.

Acknowledgements

The authors were supported by Grant FP7-People-2011-IRSES Project 295164, EUMLS: EU-Ukrainian Mathematicians for Life Sciences. The first two authors were also partially supported by the State Fund for Fundamental Research (Ukraine), Project 20570 and by Project 15-1bb\19 “Metric Spaces, Harmonic Analysis of Functions and Operators and Singular and Nonclassic Problems of Differential Equations” (Donetsk National University, Vinnytsia, Ukraine). The first author was also partially supported by the NAS of Ukraine in the frame of scientific research project for young scientists “Geometric properties of metric spaces and mappings in Finsler manifolds”.

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ОБМЕЖЕНІСТЬ КОНСТАНТ ЛЕБЕГА Й ІНТЕРПОЛЯЦІЙНІ БАЗИСИ ФАБЕРА

Проблематика. Досліджується взаємозв'язок між обмеженістю констант Лебега для поліноміальної інтерполяції Лагранжа на компакті з $\mathbb{R}$ та існуванням базису Фабера у просторі функцій, неперервних на цьому компакті.

Мета дослідження. Метою роботи є опис умов на матрицю вузлів інтерполяції, за яких інтерполювання будь-якої неперервної функції збігається з розкладанням цієї функції у ряд по базису Фабера.

Методика реалізації. Використовуються методи загальної теорії базисів Шаудера та результати, які описують збіжність інтерполяційних процесів Лагранжа.

Результати дослідження. Описано структуру матриць вузлів інтерполяції, що породжують інтерполяційні базиси Фабера.

Висновки. Кожний інтерполяційний базис Фабера породжується інтерполяційним процесом Лагранжа з матрицею спеціального виду та обмеженими константами Лебега.

Ключові слова: константа Лебега; функція Лебега; поліноміальна інтерполяція Лагранжа; базис Фабера.

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ОГРАНИЧЕННОСТЬ КОНСТАНТ ЛЕБЕГА И ИНТЕРПОЛЯЦИОННЫЕ БАЗИСЫ ФАБЕРА

Проблематика. Исследуется взаимосвязь между ограниченностью констант Лебега для полиномиальной интерполяции Лагранжа на компакте из $\mathbb{R}$ и существованием базиса Фабера в пространстве функций, непрерывных на этом компакте.

Цель исследования. Целью работы является описание условий на матрицу узлов интерполяции, при которых интерполирование любой непрерывной функции совпадает с разложением этой функции в ряд по базису Фабера.

Методика реализации. Используются методы общей теории базисов Шаудера и результаты, описывающие сходимость интерполяционных процессов Лагранжа.

Результаты исследования. Описана структура матриц узлов интерполяции, порождающих интерполяционные базисы Фабера.

Выводы. Каждый интерполяционный базис Фабера порождается интерполяционным процессом Лагранжа с матрицей интерполяции специального вида и ограниченными константами Лебега.

Ключевые слова: константа Лебега; функция Лебега; полиномиальная интерполяция Лагранжа; базис Фабера.

Рекомендована Радою
факультету прикладної математики
КПІ ім. Ігоря Сікорського

Надійшла до редакції
30 січня 2017 року